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MEMORANDUM REPORT ARBRL-MR-03335

CALCULATION OF LEGENDRE FUNCTIONS ON THE CUT FOR INTEGRAL ORDER AND COMPLEX DEGREE BY MEANS OF GAUSS CONTINUED FRACTIONS

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February 1984

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

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A method for calculating the associated Legendre functions for integral order and complex degree is presented. Comparison with special cases indicates precision of at least 25 significant digits is attained.

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I. INTRODUCTION

The associated Legendre functions on the cut for integral order and complex degree are required for several important mechanical applications. One area of particular interest is in the calculation of biharmonic functions for the hollow cone. Such functions appropriate for the cone satisfy

$$\nabla^4 \chi = 0$$

and come in the form

$$\chi=[AP_\nu^m(\cos\theta) + BQ_\nu^m(\cos\theta) + CP_{\nu-2}^m(\cos\theta) + DQ_{\nu-2}^m(\cos\theta)]\ R^\nu \sin m\phi.$$

These functions are required for stress analysis in the minihat gage, which is a strain-type pressure transducer, and in cone-plate rheometers.

We have developed a subroutine for computing

$$P_\nu^m(\cos\theta) \text{ and } Q_\nu^m(\cos\theta)$$

for

$$0<\theta\leq 90^\circ ,$$

integer m such that $0\leq m\leq 12$,

and ν such that $\mathrm{Re}(\nu)\geq -\frac{1}{2}$.

For moderate values of $|\nu|$, we calculate $P_\nu^m(\cos\theta)$ by series and recurrence formulas. On the other hand, $Q_\nu^m(\cos\theta)$ is computed through the use of a Gauss continued fraction and Wronskian relations. For small angles, the logarithmic solution is used to calculate $Q_\nu^m(\cos\theta)$. Additional analysis will be necessary for the programming of $\dfrac{\partial Q_\nu^m(\cos\theta)}{\partial\nu}$ and $\dfrac{\partial P_\nu^m(\cos\theta)}{\partial\nu}$. Asymptotic expansions are required when $|\nu|$ is large.

Programming was done on a CDC 7600 computer in FORTRAN V. CDC double precision arithmetic was used to get the greatest accuracy possible. Special

multiple precision algorithms were avoided so that the subroutine would be more compatible for use by an engineering laboratory.

II. ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

A. Hypergeometric Series

The Legendre functions of the first kind can be accurately calculated for moderate $|\nu|$ by[1]

$$P_\nu^{-m}(\cos\theta) = \frac{1}{m!} (\tan \theta/2)^m {}_2F_1(-\nu,\nu+1;1+m;(\sin\theta/2)^2) \tag{1}$$

and the reflection formula[2]

$$P_\nu^{m}(\cos\theta) = (-1)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} P_\nu^{-m}(\cos\theta) , \tag{2}$$

where

$$\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} = (\nu+1-m)(\nu+2-m)\ldots(\nu+m). \tag{2a}$$

The hypergeometric series, for small $|\nu|$, $(|\nu|<2)$, can be calculated directly from the series[3]

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots . \tag{3}$$

For larger values of $|\nu|$, the series becomes slowly convergent with serious round-off error. In this case, one can alter the parameters and then use a recurrence formula to obtain the desired results.

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1 A. Erdelyi, et. al., Higher Transcendental Functions, Vol. 1, Bateman Manuscript Project, McGraw Hill Book Co., Inc., New York, 1953.

2 W. Magnus and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics, Chelsea Publishing Co., New York, 1949.

3 M. Abramowitz and J.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Washington, D.C., No. 55, Applied Mathematics Series, June 1964.

B. Gauss Contiguous Formula

One possible method to improve the convergence of the hypergeometric series would be to increase m to a high value and then use a Legendre recurrence formula to obtain the desired answer. Unfortunately, either serious underflow or overflow problems would begin to occur in the calculation of the initial values for the Legendre recurrence formula, as found by Smith, Olver, and Lozier.[4] Occasionally both underflow and overflow occur in the same sequence of calculations. The necessary recurrence formula,[5]

$$P_{\nu}^{m+1}(\cos\theta) + 2m\cot\theta P_{\nu}^{m}(\cos\theta) + (\nu+m)(\nu-m+1)P_{\nu}^{m-1}(\cos\theta) = 0, \tag{4}$$

also contains both linear and quadratic coefficients which would lead to rapid variations of the Legendre functions in a recurrence chain.

In order to avoid the difficulties in dealing with the Legendre function, we deal directly with the hypergeometric series using the Gauss contiguous formula. In the required Gauss contiguous formula,[1]

$$\begin{aligned} &c(c-1)(z-1)F(a,b;c-1;z)+c[c-1-(2c-a-b-1)z]F(a,b;c;z) \\ &\quad +(c-a)(c-b)z\ F(a,b;c+1;z)=0 \quad , \end{aligned} \tag{5}$$

the coefficients of the hypergeometric functions are of the same degree in the parameter c. Therefore, the values of the hypergeometric functions vary slowly as the parameters change in the recurrence chain. Also, the stability criterion for the Gauss contiguous formula can be verified by elementary methods. Finally, through the use of the hypergeometric functions as the basis of our analysis, we gain an additional parameter leading to greater flexibility in deriving algorithms.

Since the Gauss contiguous formula appears to be superior for the intended work, it is used in the program. We increase c to a sufficiently large value and then increment downwards with the Gauss contiguous formula to the desired result. Using this technique has produced excellent results, provided that $|\nu|$ is not too large.

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4 *J.M. Smith, F.V.J. Olver and D.W. Lozier, "Legendre Polynomials," ACM Transactions on Mathematical Software, 7, 94-95, 1981.*

5 *A.S. Elder, "Formulas for Calculating Bessel Functions of Integral Order and Complex Argument," USA Ballistic Research Laboratory, Report No. BRL-TR-1423, 1968 AD #680209.*

III. ASSOCIATED LEGENDRE FUNCTIONS OF THE SECOND KIND

A. Gauss Continued Fraction Method

In attempting to calculate $Q_\nu^m(\cos\theta)$, no series was found which converges well for small ν and θ. Instead, an approach previously used to calculate Bessel functions of the second kind was used.[5] This approach utilizes the Gauss continued fraction and Wronskian relations. It was found to be highly advantageous to leave the cut for the initial calculations and work in the complex plane. However, $Q_\nu^m(x)$, as defined on the cut $-1<x<+1$, is not analytic when $x=z$, a complex number. We have the formula[1]

$$Q_\nu^m(x) = \tfrac{1}{2}\, e^{-im\pi}\left[e^{-\frac{1}{2}i\pi} Q_\nu^m(x+i0) + e^{\frac{1}{2}i\pi} Q_\nu^m(x-i0)\right] , \tag{6}$$

where $x+i0$, $x-i0$ indicate limits as the complex variable z approaches the cut. If $\nu = \alpha+i\beta$, and β is large,

$$Q_\nu^m(\cos\theta+i0) \text{ is } O(e^{-\beta\theta})$$

and

$$Q_\nu^m(\cos\theta-i0) \text{ is } O(e^{+\beta\theta}), \text{ where } x = \cos\theta.$$

This behavior corresponds to that of Hankel functions in the theory of Bessel functions.

In the complex plane, we have[1]

$$e^{-im\pi} Q_\nu^m(\cos\theta+i0)\Gamma(\nu+3/2) = \pi^{\frac{1}{2}}\, \Gamma(\nu+m+1) e^{\frac{1}{2}m\pi}(2\sin\theta)^m$$
$$\times\, e^{-i\theta(1+\nu+m)} F(\tfrac{1}{2}+m,\ 1+\nu+m;\ \nu+3/2;\ e^{-2i\theta}). \tag{7}$$

Replacing ν by $\nu-1$ and then dividing the result into the previous equation,

$$R_\nu^m(\cos\theta+i0) = \frac{Q_\nu^m(\cos\theta+i0)}{Q_{\nu-1}^m(\cos\theta+i0)} = \frac{(\nu+m)}{(\nu+\frac{1}{2})}\, e^{-i\theta}\, \frac{F(\frac{1}{2}+m, 1+\nu+m; \nu+3/2;\ e^{-2i\theta})}{F(\frac{1}{2}+m, \nu+m; \nu+\frac{1}{2}; e^{-2i\theta})} . \tag{8}$$

Letting $a = \frac{1}{2} + m, b = \nu + m, c = \nu + \frac{1}{2}, z = e^{-2i\theta}$,

$$R_\nu^m(\cos\theta + i0) = \frac{b\ e^{-i\theta}}{c} \ \frac{F(a,b+1;\ c+1;\ z)}{F(a,\ b;\ c;\ z)} \ . \tag{9}$$

This can be calculated by using the Gauss continued fraction[6]

$$\frac{F(a,b+1;c+1;z)}{F(a,b;c;z)} = \cfrac{1}{1 - \cfrac{\frac{a(c-b)z}{c(c+1)}}{1 - \cfrac{\frac{(b+1)(c-a+1)z}{(c+1)(c+2)}}{1 - \cfrac{\frac{(a+1)(c-b+1)z}{(c+2)(c+3)}}{1 - \cfrac{\frac{(b+2)(c-a+2)z}{(c+3)(c+4)}}{1 - \cdots}}}}} \ . \tag{10}$$

The Gauss continued fraction converges everywhere in the complex plane, except on the real axis from +1 to $+\infty$ and at isolated zeros of $F(a,b;c;z)$.

In order to obtain $Q_\nu^m(\cos\theta + i0)$ from its continued fraction, we utilize the following Wronskian relations[2]:

$$W\{P_\nu^{-m}(z),\ Q_\nu^m(z)\} = P_\nu^{-m}(z)\ \frac{dQ_\nu^m(z)}{dz} - Q_\nu^m(z)\ \frac{dP_\nu^{-m}(z)}{dz} \tag{11}$$

and

$$W\{P_\nu^{-m}(z),\ Q_\nu^m(z)\} = \frac{e^{i\pi m}}{(1-z^2)} \ .$$

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[6] *H.S. Wall, Analytic Theory of Continued Fractions, d. Van Nostrand Company, Inc., New York, 1948.*

Equating these, we have

$$P_\nu^{-m}(z) \frac{dQ_\nu^m(z)}{dz} - Q_\nu^m(z) \frac{dP_\nu^{-m}(z)}{dz} = \frac{e^{i\pi m}}{(1-z^2)} . \qquad (12)$$

Now using the derivative relations[1]

$$(z^2-1) \frac{dP_\nu^{-m}(z)}{dz} = \nu z \; P_\nu^{-m}(z) - (\nu-m) \; P_{\nu-1}^{-m}(z) \qquad (13)$$

and

$$(z^2-1) \frac{dQ_\nu^m(z)}{dz} = \nu z \; Q_\nu^m(z) - (\nu+m) \; Q_{\nu-1}^m(z) ,$$

we find

$$(\nu+m) \; P_\nu^{-m}(z) \; Q_{\nu-1}^m(z) - (\nu-m) \; P_{\nu-1}^{-m}(z) \; Q_\nu^m(z) = e^{i\pi m} . \qquad (14)$$

Dividing both sides by $Q_{\nu-1}^m(z)$ and replacing z by $\cos\theta+i0$, we get

$$(\nu+m) \; P_\nu^{-m}(\cos\theta+i0) - (\nu-m) \; P_{\nu-1}^{-m}(\cos\theta+i0) \; R_\nu^m(\cos\theta+i0) = \frac{e^{i\pi m}}{Q_{\nu-1}^m(\cos\theta+i0)} .$$

$$(15)$$

Replacing ν by $\nu+1$ and then solving for $Q_\nu^m(\cos\theta+i0)$,

$$Q_\nu^m(\cos\theta+i0) = \frac{e^{i\pi m}}{[(\nu+m+1)P_{\nu+1}^{-m}(\cos\theta+i0)-(\nu-m+1)P_\nu^{-m}(\cos\theta+i0)R_{\nu+1}^m(\cos\theta+i0)]} .$$

$$(16)$$

Since[1]

$$P_\nu^{-m}(\cos\theta+i0) = e^{1/2\, im\pi}\, P_\nu^{-m}(\cos\theta)\ , \tag{17}$$

$$Q_\nu^m(\cos\theta+i0) = \frac{e^{1/2\, im\pi}}{[(\nu+m+1)P_{\nu+1}^{-m}(\cos\theta)-(\nu-m+1)P_\nu^{-m}(\cos\theta)R_{\nu+1}^m(\cos\theta+i0)]}\ . \tag{18}$$

Now we can use the relation[1]

$$Q_\nu^m(\cos\theta) = e^{3/2\ im\pi}\ Q_\nu^m(\cos\theta+i0) + i\pi/2\ P_\nu^m(\cos\theta) \tag{19}$$

to finally get

$$Q_\nu^m(\cos\theta) = \frac{(-1)^m}{[(\nu+m+1)P_{\nu+1}^{-m}\ (\cos\theta)-(\nu-m+1)P_\nu^{-m}(\cos\theta)R_{\nu+1}^m(\cos\theta+i0)]} + i\pi/2\ P_\nu^m(\cos\theta). \tag{20}$$

Results from programming the above method indicate that it works exceptionally well. Unfortunately, this method will likely run into difficulties for large $|\nu|$. Even though the Gauss continued fraction seems to converge more rapidly with increasing $|\nu|$, difficulties may be encountered because of losses in accuracy with $P_\nu^m(\cos\theta)$. For large $|\nu|$, it will be necessary to use asymptotic expansions to calculate $Q_\nu^m(\cos\theta)$ and $P_\nu^m(\cos\theta)$.

Another region where trouble occurs is at small angles ($\theta<1^o$). The complex Gauss continued fraction algorithm is still very accurate at small angles, but the continued fraction converges at a very slow rate. Sometimes several thousand terms are required for convergence, which in turn uses an inordinate amount of computer time. This problem occurs because the argument approaches the cut of the Gauss continued fraction at +1. To avoid this problem, we use the logarithmic solution to calculate $Q_\nu^m(\cos\theta)$ for small angles.

A similar method for computing $Q_\nu^m(\cos\theta)$ has also been developed using real analysis only. Since no simple series formula for $Q_\nu^m(\cos\theta)$ could be found, we concentrated on $P_\nu^m(-\cos\theta)$ instead. $P_\nu^m(-\cos\theta)$ is related to $Q_\nu^m(\cos\theta)$ and $P_\nu^m(\cos\theta)$ by[1]

$$Q_\nu^m(\cos\theta) = \frac{\pi}{2\sin[\pi(\nu+m)]}\left\{\cos[\pi(\nu+m)]P_\nu^m(\cos\theta) - P_\nu^m(-\cos\theta)\right\},\ 0<\theta<\pi/2. \tag{21}$$

Similar computational difficulties exist for the computation of $P_\nu^m(-\cos\theta)$ as $Q_\nu^m(\cos\theta)$. Using the same equations as in calculating $P_\nu^m(\cos\theta)$, we can derive the following continued fraction

$$\frac{P_\nu^m(-\cos\theta)}{P_{\nu-1}^m(-\cos\theta)} = \frac{(\nu+m)}{(\nu-m)}\ \frac{F(-\nu,\nu+1;\ 1+m;\ (\cos\theta/2)^2)}{F(-\nu+1,\ \nu;\ 1+m;\ (\cos\theta/2)^2)}\ . \tag{22}$$

Letting $a=-\nu$, $b=\nu$, $c=m$, $z=(\cos\theta/2)^2$, then

$$S_\nu^m(z) = \frac{F(a,b+1;c+1;z)}{F(a+1,b;c+1;z)} = \frac{F(a,b+1;c+1;z)}{F(a,b;c;z)}\ \frac{F(b,a;c;z)}{F(b,a+1;c+1;z)}\ . \tag{23}$$

This can be calculated by either two Gauss continued fractions or by two continued fractions of Frank[7] which are of the form

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} =$$

(Equation is incomplete, continued on next page)

[7] *E. Frank, "A New Class of Continued Fraction Expansions for the Ratios of Hypergeometric Functions," Transactions of the American Mathematical Society,* 81 , 454, 1956.

$$1 - \cfrac{\frac{a(c-b)z}{c(c+1)}}{\frac{(a-b)z}{(c+1)} + 1 - \cfrac{\frac{(a+1)(c+1-b)z}{(c+1)(c+2)}}{\frac{(a+1-b)z}{(c+2)} + 1 - \cfrac{\frac{(a+2)(c+2-b)z}{(c+2)(c+3)}}{\frac{(a+2-b)z}{(c+3)} + 1 - \dots}}} \quad . \tag{24}$$

The continued fraction of Frank has its cut on the unit circle. It has proved to be of little value here because of its much slower convergence rate as compared to the Gauss continued fraction. This is especially true for small angles, which effectively limit the use of the Frank continued fraction to angles greater than 30°.

Utilizing similar Wronskian relations as in the complex analysis, we proceeded in a similar manner, with the exception of replacing $Q_{\nu}^{m}(\cos\theta)$ by Eq. (21). The final result of the real analysis is then

$$Q_{\nu}^{m}(\cos\theta) = \frac{\pi}{2} P_{\nu}^{m}(\cos\theta) \cot[\pi(\nu+m)] + \frac{(-1)^{m}}{(\nu+1+m)[P_{\nu+1}^{-m}(\cos\theta) + P_{\nu}^{-m}(\cos\theta) S_{\nu+1}^{m}(z)]} . \tag{25}$$

This result can easily be seen to be inferior to the complex analysis since it is undefined for integral ν . Also, it is necessary to calculate two continued fractions which have relatively slow convergence rates which worsen with increasing $|\nu|$. Therefore, only the complex algorithm is used in the final programming. The real method will be useful in helping to check and compare the accuracy of the complex method for calculating $Q_{\nu}^{m}(\cos\theta)$.

B. The Logarithmic Solution

As mentioned previously, the logarithmic solution is used for the calculation of $Q_{\nu}^{m}(\cos\theta)$ at small angles ($\theta<1^{\circ}$). In this range, the

logarithmic solution is as accurate as the Gauss continued fraction method, but requires much less computer time to obtain results. From references [1] and [2], the following formulas can be derived.

$$Q_\nu^m(\cos\theta) = \tfrac{1}{2}\, P_\nu^m(\cos\theta)\left[\log(1/\tan^2(\theta/2)) - 2\gamma - 2\psi(\nu+1) + \sum_{s=1}^{m} \frac{2s-1}{(\nu+s)(\nu+1-s)}\right]$$

$$+ \frac{(-1)^m}{2\nu} \left(\frac{1}{\tan\theta/2}\right)^m \sum_{r=0}^{m} (-\nu^2)(1-\nu^2)(4-\nu^2)\ldots((r-1)^2-\nu^2)(\nu+r)\frac{(m-r-1)!}{r!}$$

$$\times (-1)^r(\sin\theta/2)^{2r}$$

$$+ \frac{(\sin\theta/2)^m}{2\nu} \sum_{\ell=1}^{\infty} \frac{(-\nu^2)(1-\nu^2)\ldots((m+\ell-1)^2-\nu^2)(m+\nu+\ell)}{\ell!(m+\ell)!}$$

$$\times \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\ell}\right)(\sin\theta/2)^{2\ell}$$

$$+ \frac{(-1)^m}{2\nu} (\nu-m+1)(\nu-m+2)\ldots(\nu+m)\left(\tan\theta/2\right)^m$$

$$\times \sum_{r=0}^{\infty} \frac{(-\nu)^2(1-\nu^2)\ldots((r-1)^2-\nu^2)(\nu+r)}{\nu!\ \ (r+m)!} \left(1+ \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{m+r}\right)(\sin\theta/2)^{2r} \tag{26}$$

For $m=0$, but ν not an integer,

$$Q_\nu^o(\cos\theta) = \tfrac{1}{2}\, P_\nu^o(\cos\theta)\, [\log(\cot^2(\theta/2)) - 2\gamma - 2\psi(\nu+1)]$$

$$+ \frac{1}{\nu} \sum_{\ell=1}^{\infty} \frac{(-\nu^2)(1-\nu^2)\ldots((\ell-1)^2-\nu^2)(\ell+\nu)}{(\ell!)^2} \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{\ell}\right)(\sin\theta/2)^{2\ell} .$$

(26a)

For $m=0$ and $\nu=n$, an integer,

$$Q_n^o(\cos\theta) = \tfrac{1}{2}\, P_n^o(\cos\theta)\, [\log(\cot^2(\theta/2)) - 2(1+\frac{1}{2}+\frac{1}{3}+\ldots 1/n)]$$

$$+ \sum_{\ell=1}^{n} \frac{(-1)^\ell (n+\ell)!\ (1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{\ell})}{(\ell!)^2 (n-\ell)!} (\sin\theta/2)^{2\ell} . \qquad (26b)$$

IV. CONCLUSIONS

We have developed an accurate and efficient method of calculating the associated Legendre functions for integral order and complex degree. In general, the accuracy of $P_\nu^m(\cos\theta)$ and $Q_\nu^m(\cos\theta)$ was approximately equal to the maximum attainable with double precision arithmetic. By comparisons with special cases ($P^m_{-1/2+i\lambda}(\cos\theta)$, $Q_\nu^m(\cos\theta)$ for real ν, Legendre Polynomials) and alternate means of calculation of the associated Legendre functions (Gaussian quadratures, alternate series), the modulus of the results appeared accurate

to at least 25 significant figures. Tabular and graphical results are presented in Appendix A and Appendix B, respectively.

In routines still under development at BRL, special series and nonhomogeneous recurrence formulas are being used to calculate $\frac{\partial P_\nu^m(\cos\theta)}{\partial \nu}$ and $\frac{\partial Q_\nu^m(\cos\theta)}{\partial \nu}$. Also being programmed are asymptotic expansions to extend the range of $|\nu|$ to very large values. For small angles ($\theta < 30^0$), Bessel function expansions will be used.[8] Special hypergeometric series along with sines and cosines will be used for larger angles.

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[8] *N.K. Chu Khrukidze, "Asymptotic Formulae for Legendre Functions," USSR Computational Mathematics and Mathematical Physics, 6, 1966.*

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5. A.S. Elder, "Formulas for Calculating Bessel Functions of Integral Order and Complex Argument," USA Ballistic Research Laboratory, Report No. BRL-TR-1423, 1968 AD# 680209.

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APPENDIX A

NUMERICAL CALCULATIONS

LEGENDRE FUNCTION OF THE FIRST KIND
COMPLEX DEGREE
OF ORDER M=3

$P^{3}_{-\frac{1}{2}+i\tau}(.5)$

T°	RE(P)	IM(P)	|P|
0.0	-.42625D+02	0.	.42624687842515339645402000D+02
5.0	-.49585D+02	.64943D+01	.50008347609597419739303240D+02
10.0	-.70211D+02	.52977D+01	.70410251196144400103116260D+02
15.0	-.10209D+03	-.13976D+02	.10304349357104659429312100D+03
20.0	-.13579D+03	-.64986D+02	.15053858308326658261820150D+03
25.0	-.14791D+03	-.15994D+03	.21784848452045324799895990D+03
30.0	-.96827D+02	-.29600D+03	.31143066791158882088248800D+03
35.0	.68674D+02	-.43319D+03	.43860221502933838260313520D+03
40.0	.37572D+03	-.47622D+03	.60658910001679639331638240D+03
45.0	.76870D+03	-.28849D+03	.82105029740159984855906490D+03
50.0	.10593D+04	.23040D+03	.10840842067934589168622050D+04
55.0	.95627D+03	.10115D+04	.13919838224765219311152190D+04
60.0	.23330D+03	.17175D+04	.17332876847308086674211650D+04
65.0	-.10143D+04	.18250D+04	.20878779034119939274684600D+04
70.0	-.22321D+04	.95508D+03	.24278642845138677923181747D+04
75.0	-.26230D+04	-.72251D+03	.27206584494183120291881170D+04
80.0	-.17002D+04	-.23912D+04	.29340006593244211545752910D+04
85.0	.22351D+03	-.30337D+04	.30419444970230033969349005D+04
90.0	.21091D+04	-.21758D+04	.30302544472741789175777980D+04

LEGENDRE FUNCTION OF THE FIRST KIND
COMPLEX DEGREE
OF ORDER M=10

$P^{10}_{5\epsilon^{i\tau}}$ (.5)

T°	RE(P)	IM(P)	!P!
0.0	-.17567D-23	0.	.1756740645200178556531559D-23
5.0	-.45718D+04	.86533D+04	.9786721832265384104119634D+04
10.0	-.31413D+05	.17643D+05	.3602819611507280380813874D+05
15.0	-.11283D+06	-.17059D+05	.1141137833790277162911911D+06
20.0	-.20883D+06	-.25391D+06	.3287604834674407680337729D+06
25.0	.87599D+05	-.86209D+06	.8665313218573185292398078D+06
30.0	.16911D+07	-.12402D+07	.2097066261933928173429852D+07
35.0	.45382D+07	.11287D+07	.4676464483245562325317096D+07
40.0	.39000D+07	.88166D+07	.9640610184075392214905948D+07
45.0	-.92109D+07	.15954D+08	.1842179051579653261948696D+08
50.0	-.32588D+08	.26579D+07	.3269598814179090519596817D+08
55.0	-.33264D+08	-.42515D+08	.5398146578923364656372896D+08
60.0	.27972D+08	-.78135D+08	.8299103461893121083518568D+08
65.0	.11645D+09	-.23948D+08	.1188892059351971589462901D+09
70.0	.10655D+09	.11769D+09	.1587622518614721515430449D+09
75.0	-.60836D+08	.18807D+09	.1976650733531009581474026D+09
80.0	-.22482D+09	.45924D+08	.2294661653843174273698999D+09
85.0	-.16208D+09	-.18820D+09	.2483746055683535114110683D+09
90.0	.86459D+08	-.23527D+09	.2506514797722299392813258D+09

LEGENDRE FUNCTION OF THE SECOND KIND
COMPLEX DEGREE
OF ORDER M=5

$Q^{5}_{5(iT}(.5)$

T	RE(Q)	IM(Q)	!Q!
57.50	-.18966D+05	-.27949D+05	.33776251385318829792752810D+05
57.55	-.54764D+04	-.39093D+05	.39474723994468911539028680D+05
57.60	.14433D+05	-.43173D+05	.45521237126130919936229490D+05
57.65	.36754D+05	-.36491D+05	.51792740815031983821134870D+05
57.70	.55339D+05	-.17805D+05	.58133151683613064821378250D+05
57.75	.63503D+05	.10439D+05	.64355388963556668994804760D+05
57.80	.56331D+05	.41970D+05	.70247203980715377923883410D+05
57.85	.32901D+05	.68044D+05	.75581268279396651032496650D+05
57.90	-.25558D+04	.80088D+05	.80129267313383237769359590D+05
57.95	-.41283D+05	.72786D+05	.83678895859810127956851070D+05
58.00	-.72473D+05	.46397D+05	.86051872107269658884209020D+05
58.05	-.86820D+05	.72268D+04	.87120589428358254946585150D+05
58.10	-.79831D+05	-.34131D+05	.86820962370953959369312920D+05
58.15	-.53505D+05	-.66252D+05	.85159449722613237042031150D+05
58.20	-.15683D+05	-.80703D+05	.82213087359659378219558800D+05
58.25	.22724D+05	-.74745D+05	.78122466852189489111992370D+05
58.30	.51472D+05	-.51876D+05	.73078718534435249192152720D+05
58.35	.64218D+05	-.20154D+05	.67306460096055501356699880D+05
58.40	.60065D+05	.10897D+05	.61045167694305196344973680D+05
58.45	.43049D+05	.33474D+05	.54531430237510680199921980D+05
58.50	.20050D+05	.43594D+05	.47984093324010848496556470D+05

LEGENDRE FUNCTION OF THE SECOND KIND
COMPLEX DEGREE
OF ORDER M=10

$Q^{i\tau}_{5e^{i\tau}}(.5)$

T	RE(Q)	IM(Q)	!Q!
57.50	-.30838D+08	.80986D+08	.86658925048089391200111910D+08
57.55	-.76502D+08	.42454D+08	.87492162511892137343823230D+08
57.60	-.97126D+08	-.30047D+08	.10166766316849155072568460D+09
57.65	-.67013D+08	-.11757D+09	.13532406169315850512956030D+09
57.70	.23620D+08	-.18191D+09	.18343381453010023622381960D+09
57.75	.15454D+09	-.17993D+09	.23718489939890022369383050D+09
57.80	.27473D+09	-.87713D+08	.28839009648586381185244170D+09
57.85	.32117D+09	.78045D+08	.33051554512426330595387810D+09
57.90	.25152D+09	.25686D+09	.35950113938522486646233413D+09
57.95	.73399D+08	.36727D+09	.37453696427176522458442630D+09
58.00	-.14957D+09	.34743D+09	.37825958512760432929727530D+09
58.05	-.32340D+09	.19140D+09	.37579383158555985312108070D+09
58.10	-.37001D+09	-.41902D+08	.37237249959895961253896250D+09
58.15	-.26828D+09	-.25533D+09	.37036077559436067011697660D+09
58.20	-.65276D+08	-.36197D+09	.36780875146626281005635600D+09
58.25	.14912D+09	-.32752D+09	.35987102086249545399941700D+09
58.30	.28871D+09	-.18294D+09	.34178548660998129914610150D+09
58.35	.31121D+09	-.23110D+07	.31122215027024302417296080D+09
58.40	.23110D+09	.13795D+09	.26914246847109541850294700D+09
58.45	.10224D+09	.19421D+09	.21948043609320690694917710D+09
58.50	-.14857D+08	.16760D+09	.16826175465375806929697130D+09

LEGENDRE FUNCTION OF THE SECOND KIND
REAL DEGREE
OF ORDER M=5

Q_{ν}^{5} (.5)

V	RE(Q)	IM(Q)	!Q!
.10	-.18901D+03	-.21331D-26	.18900801878022575115073688D+03
.50	-.19544D+03	-.51244D-26	.19544238489555303792612610D+03
1.00	-.20939D+03	-.34167D-26	.20938569762610427726198466D+03
1.50	-.23388D+03	-.13278D-25	.23387516642480665348073920D+03
2.00	-.27097D+03	-.52171D-26	.27096972633966435880962720D+03
2.50	-.32105D+03	.63614D-26	.32105006758853366329468730D+03
3.00	-.39414D+03	-.97892D-26	.39413778376678452190491220D+03
5.00	-.18502D+04	.29727D-23	.18502430608177084179696150D+04
10.00	.29655D+04	-.33584D-21	.29654539432231117608705680D+04
15.00	.28410D+06	-.50293D-21	.28409853164708937102225370D+06
20.00	.60375D+06	.82850D-20	.60375371292282375720737980D+06
25.00	-.13587D+07	.25305D-19	.13587387943672703353943740D+07
30.00	-.63491D+07	-.23889D-20	.63490511712187315497813710D+07
40.00	.13079D+08	-.36846D-18	.13078752708656431332812410D+08
50.00	.24329D+08	.12265D-17	.24328639125798572084814930D+08

LEGENDRE FUNCTION OF THE SECOND KIND
REAL DEGREE
OF ORDER M=10

Q_{V}^{10} (.5)

V	RE(Q)	IM(Q)	!Q!
.10	.44220D+08	.78601D-21	.4422019441063084063027027D+08
.50	.44992D+08	.83915D-21	.4499152949357942571172876D+08
1.00	.46540D+08	-.11364D-20	.4654023111111111111111111D+08
1.50	.48805D+08	.58831D-21	.4880508920090121938912310D+08
2.00	.51896D+08	.26140D-20	.5189632000000000000000000D+08
2.50	.55972D+08	.10564D-20	.5597232143658146092088463D+08
3.00	.61249D+08	.15127D-20	.6124885333333333333333333D+08
5.00	.10245D+09	.36158D-20	.1024546133333333333333333D+09
10.00	.19325D+10	-.16534D-17	.1932505390653060265343826D+10
15.00	-.14338D+12	.32786D-15	.1433781263803476703462301D+12
20.00	-.55354D+12	-.23648D-13	.5535387942959007654405108D+12
25.00	.23167D+14	-.87930D-13	.2316748891666197413966697D+14
30.00	.95826D+14	.14673D-11	.9582603845208352062519367D+14
40.00	-.23100D+16	-.26574D-11	.2309968419476341425267896D+16
50.00	.12973D+17	-.28194D-09	.1297283270469044388418573D+17

CONICAL LEGENDRE FUNCTION OF THE FIRST KIND
COMPLEX DEGREE
OF ORDER M=1

$P^{1}_{-\frac{1}{2}+iy}(.5)$

Y	RE(P)	IM(P)	!P!
.1	.15552D+00	-.61344D-37	.155520072675136542630232 5D+00
.5	.30890D+00	-.78861D-47	.3088993114335359644186385D+00
1.0	.85249D+00	-.71766D-47	.8524933343478348382398572D+00
1.5	.19973D+01	.46635D-35	.1997326275168119029587221D+01
2.0	.41877D+01	0.	.4187708475831689682782891D+01
2.5	.82464D+01	-.64443D-32	.8246410482142874879691898D+01
3.0	.15651D+02	.38005D-33	.1565137983679778097385552D+02
5.0	.17143D+03	-.73204D-30	.1714349478205223384894742D+03
10.0	.46781D+05	-.82339D-27	.4678102294059967987680579D+05
15.0	.10853D+08	-.12699D-25	.1085296077634717700984421D+08
20.0	.23641D+10	-.26393D-21	.2364063883735811264896754D+10
25.0	.49781D+12	-.20828D-19	.4978136688996549478661809D+12
30.0	.10263D+15	.89905D-17	.1026296080737662738917241D+15
40.0	.41925D+19	.81836D-12	.4192510395381910780386026D+19
50.0	.16570D+24	-.35451D-07	.1657040142902841257370712D+24

CONICAL LEGENDRE FUNCTION OF THE FIRST KIND
COMPLEX DEGREE
OF ORDER M=10

$P^{10}_{-1/2+iY}(.5)$

Y	RE(P)	IM(P)	!P!
.1	.48914D+03	-.94010D-26	.48913585134593560012170610+03
.5	.11478D+04	-.29312D-43	.11478410243829040410861450+04
1.0	.50070D+04	-.42151D-43	.50070386186075993238050600+04
1.5	.21858D+05	.51036D-31	.21857825018393532089200930+05
2.0	.92075D+05	0.	.92075211208276866556112640+05
2.5	.37393D+06	-.29222D-27	.37392953232534501245459720+06
3.0	.14652D+07	.35579D-28	.14652154098920242714762300+07
5.0	.24696D+09	-.10545D-23	.24695549286117219972770460+09
10.0	.13723D+14	-.24154D-18	.13723468018111848115909750+14
15.0	.12883D+18	-.15074D-15	.12882834633792463793109230+18
20.0	.43369D+21	.73898D-08	.43369083034979352196541640+21
25.0	.77904D+24	-.12913D-04	.77904260498089933257409090+24
30.0	.92526D+27	.22346D-01	.92526360969238511960080180+27
40.0	.59089D+33	-.88594D+04	.59089366337138470757923660+33
50.0	.19383D+39	.11264D+10	.19382736426993325063842310+39

CONICAL LEGENDRE FUNCTION OF THE SECOND KIND
COMPLEX DEGREE
OF ORDER M=1

$Q^{1}_{-1/2+i\gamma}(.5)$

Y	RE(Q)	IM(Q)	|Q|
.1	-.77264D+00	-.74317D-01	.77620144494795149481902226D+00
.5	-.70177D+00	-.44502D+00	.83097629350916298288316260D+00
1.0	-.54069D+00	-.13341D+01	.14395050700718725453807540D+01
1.5	-.38026D+00	-.31369D+01	.31598501337947940203914340D+01
2.0	-.25513D+00	-.65780D+01	.65829369379674706425735450D+01
2.5	-.16671D+00	-.12953D+02	.12954500119931210979358670D+02
3.0	-.10712D+00	-.24585D+02	.24585363008523722596962800D+02
5.0	-.16662D-01	-.26929D+03	.26928938683620172809616360D+03
10.0	-.12310D-03	-.73483D+05	.73483458998801770445147940D+05
15.0	-.61543D-07	-.17048D+08	.17047790922334920422262420D+08
20.0	.52328D+01	-.37135D+10	.37134628625998169909647760D+10
25.0	.44549D+08	-.78194D+12	.78194189502694445116166370D+12
30.0	.86531D+14	.64910D+14	.10817048658037239681947290D+15
40.0	.49804D+10	.65856D+19	.65855799268166945812382480D+19
50.0	.84420D+05	.26029D+24	.26028725698234736290819950D+24

CONICAL LEGENDRE FUNCTION OF THE SECOND KIND
COMPLEX DEGREE
OF ORDER M=10

$Q^{10}_{-1/2+iY}(.5)$

Y	RE(Q)	IM(Q)	!Q!
.1	.43782D+08	-.23374D+03	.43781989399457307721149980D+08
.5	.43500D+08	-.16536D+04	.43499831939445316981130020D+08
1.0	.42631D+08	-.78357D+04	.42630825762227396189045435D+08
1.5	.41224D+08	-.34329D+05	.41224355237761774941548300D+08
2.0	.39340D+08	-.14463D+06	.39339978251706168803719860D+08
2.5	.37054D+08	-.58737D+06	.37058171837997440309511730D+08
3.0	.34454D+08	-.23016D+07	.34530799179481372592288350D+08
5.0	.22769D+08	-.38792D+09	.38858442705555205139822480D+09
10.0	.38404D+07	-.21557D+14	.21556773153737672893926180D+14
15.0	.29287D+06	-.20236D+18	.20236309321467289828164950D+18
20.0	-.70074D+11	-.68124D+21	.68123996330234816908619560D+21
25.0	-.69402D+19	-.12237D+25	.12237197265179322867840160D+25
30.0	-.99656D+27	-.15521D+28	.18444759122757150914669060D+28
40.0	-.27078D+25	.92817D+33	.92817359724582902151311800D+33
50.0	-.31558D+21	.30446D+39	.30446331182754753912411440D+39

LEGENDRE FUNCTION OF THE SECOND KIND
OF REAL DEGREE V=2.
AND ORDER M=0

Q_2^0 [COS(T)]

T°	COMPLEX GAUSS CONTINUED FRACTION METHOD	SPECIAL CASE (FOR ACCURACY CHECK) Q_2^0[COS(T)]=LOG[COT^2(T/2)] *(3COS^2(T)-1)/4 - 3/2*COS(T)
.01	.78465439248286465236100051D+01	.78465439248286465236100051D+01
.10	.55439290837204905215287670D+01	.55439290837204905215287670D+01
.50	.39339513181244247402367970D+01	.39339513181244247402367970D+01
1.00	.32394109914617120582345970D+01	.32394109914617120582345970D+01
10.00	.84884171323235705364331520D+00	.84884171323235705364331520D+00
20.00	.21368715888553956295555630D-01	.21368715888553956295555630D-01
30.00	-.47593942009864752725493080D+00	-.47593942009864752725493080D+00
40.00	-.76476839707265500812595540D+00	-.76476839707265500812595540D+00
50.00	-.87281240461756544130134930D+00	-.87281240461756544130134930D+00
60.00	-.81866326804175685571220280D+00	-.81866326804175685571220280D+00
70.00	-.62868691870080213211947480D+00	-.62868691870080213211947480D+00
80.00	-.34025057730173553827142220D+00	-.34025057730173553827142220D+00

LEGENDRE FUNCTION OF THE SECOND KIND
REAL DEGREE
OF ORDER M=0

Q_2^0 [COS(T)]

T	LOGARITHMIC SOLUTION	SPECIAL CASE (FOR ACCURACY CHECK) Q_2^0[COS(T)]=LOG[COT^2(T/2)] *(3COS^2(T)-1)/4 - 3/2*COS(T)
.001	.1014912941946317922184000D+02	.1014912941946317922184000D+02
.005	.8539691402540814834167413D+01	.8539691402540814834167413D+01
.010	.7846543924828646523609732D+01	.7846543924828646523609732D+01
.050	.6237098088637495570857967D+01	.6237098088637495570857967D+01
.100	.5543929083720490521528448D+01	.5543929083720490521528448D+01
.250	.4627506207003646750005995D+01	.4627506207003646750005995D+01
.500	.3933951318124424740236479D+01	.3933951318124424740236479D+01
.750	.3527877952305230389869642D+01	.3527877952305230389869642D+01
1.000	.3239410991461712058234278D+01	.3239410991461712058234278D+01
5.000	.1601330656918337463046403D+01	.1601330656918337463046403D+01
10.000	.8488417132323570536429548D+00	.8488417132323570536429548D+00

LEGENDRE FUNCTION OF THE SECOND KIND
OF DEGREE V=.1
OF ORDER M=1

$Q_{.1}^{1}$ (COS(T))

T°	LOGARITHMIC SOLUTION	COMPLEX GAUSS CONTINUED FRACTION METHOD
.005	-.11459155967011203778918480+05	-.11459155967011203778918480+05
.010	-.57295780734439762339710660+04	-.57295780734439762339710660+04
.050	-.11459161236929236694631360+04	-.11459161236929236694631360+04
.100	-.57295879545620491117545330+03	-.57295879545620491117545330+03
.250	-.22918539897471327138320970+03	-.22918539897471327138320970+03
.500	-.11459578820359369037168810+03	-.11459578820359369037168810+03
.750	-.76400424582515658591572750+02	-.76400424582515658591572750+02
1.000	-.57303572639238578099893150+02	-.57303572639238578099893150+02
5.000	-.11490416444659423566983750+02	-.11490416444659423566983750+02
10.000	-.57855544318912687421605l0+01	-.57855544318912687421605l0+01

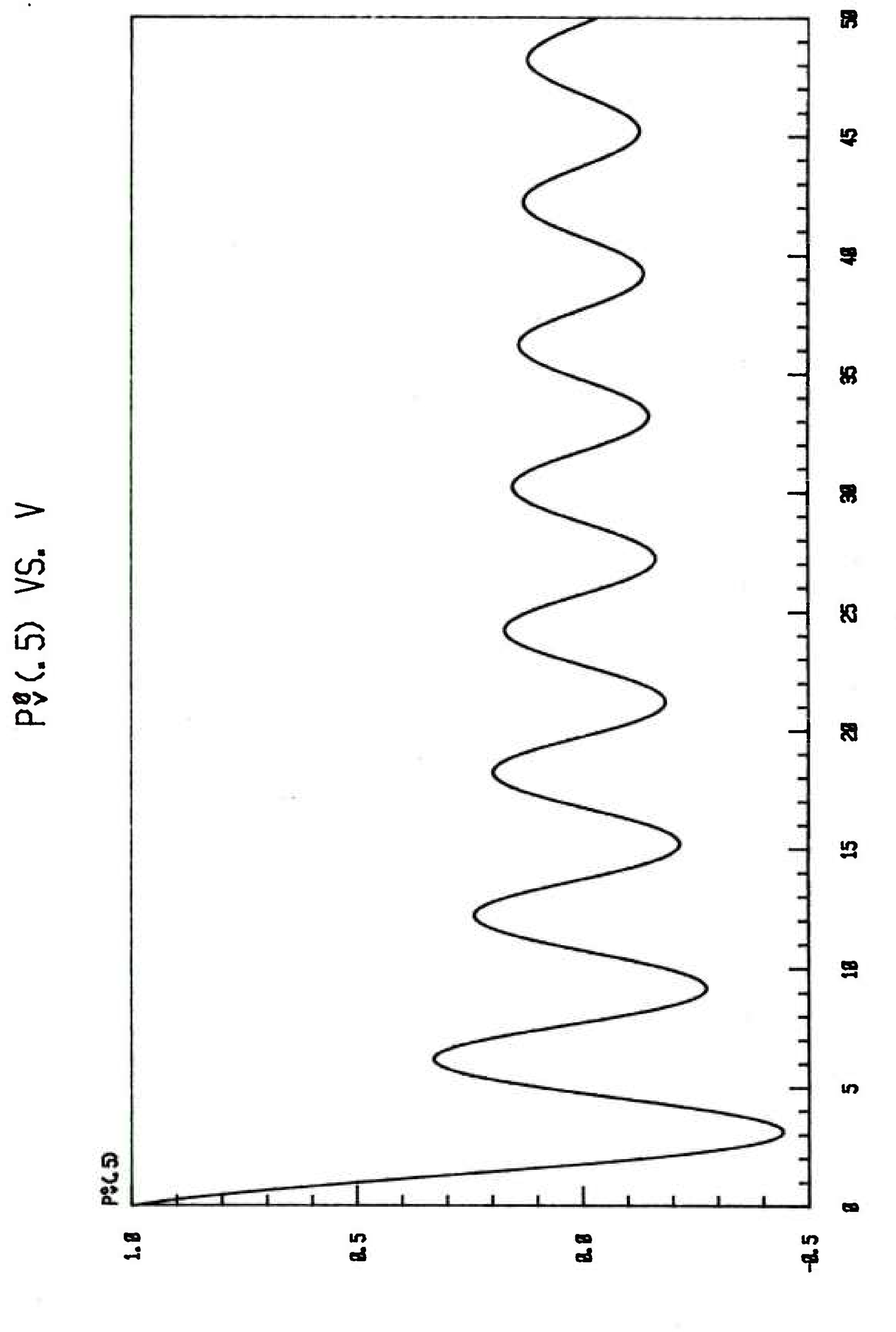
LEGENDRE FUNCTION OF THE SECOND KIND
OF DEGREE V=10.5
OF ORDER M=1

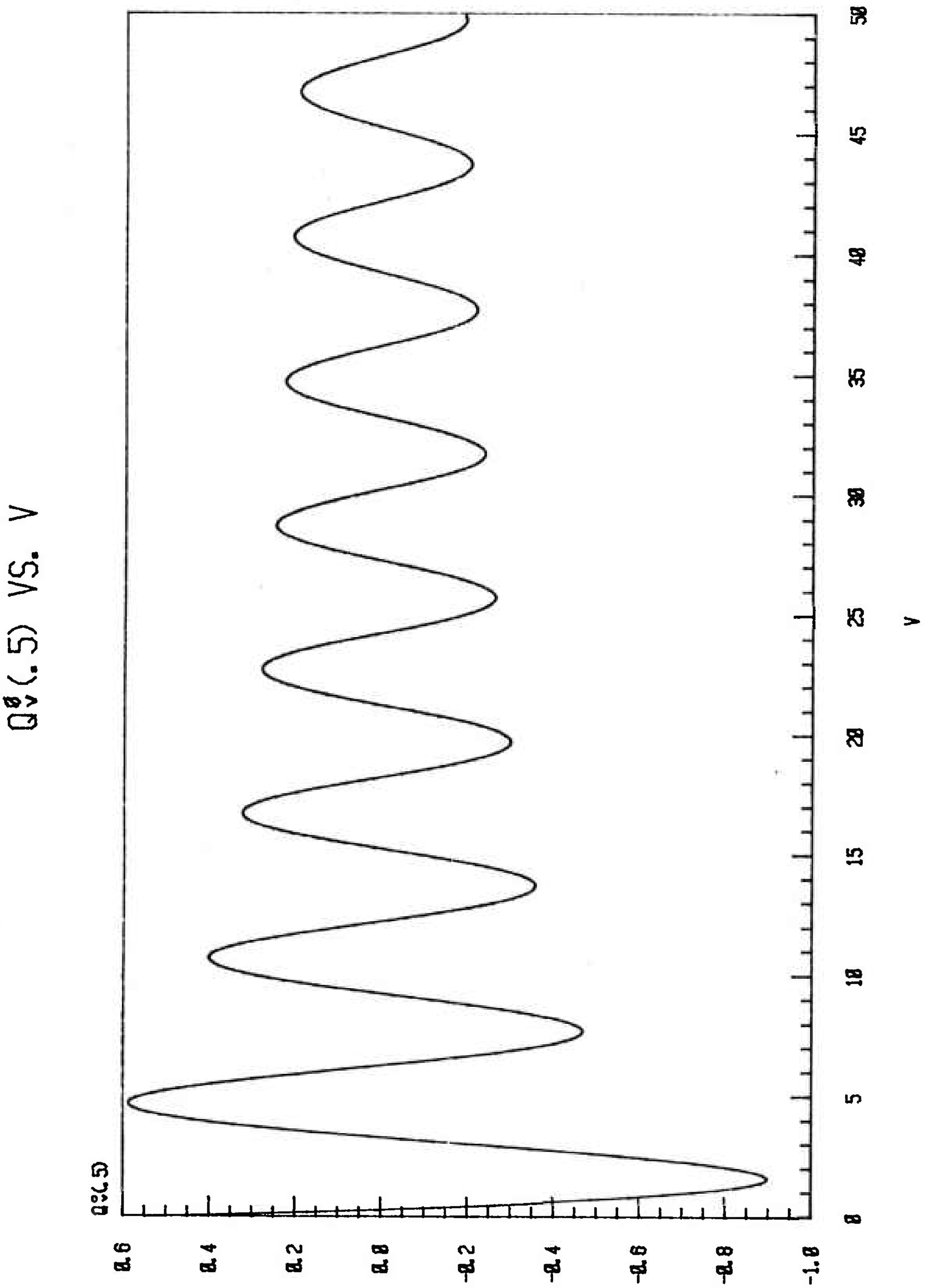
$Q^{1}_{10.5}$[COS(T)]

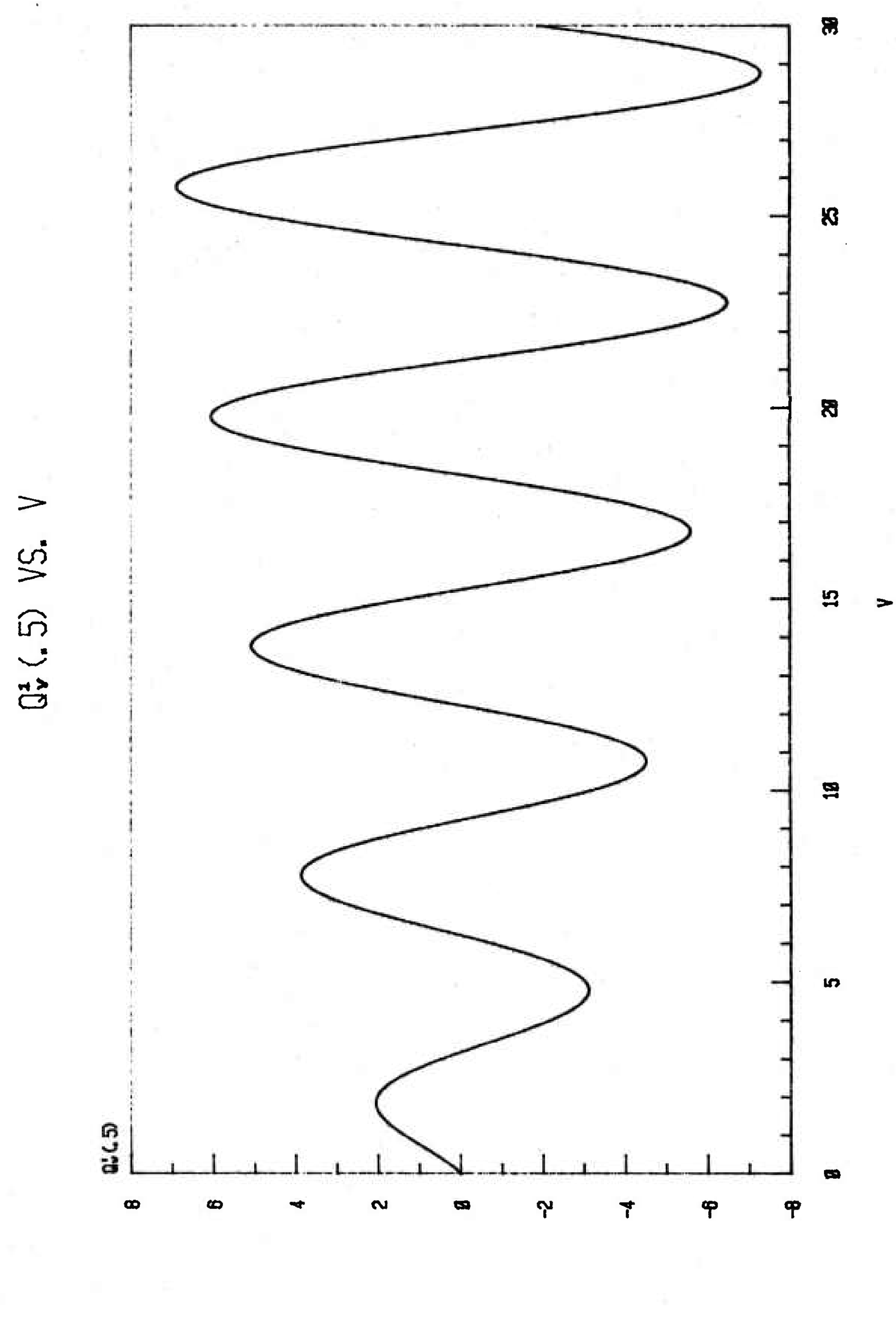
T°	LOGARITHMIC SOLUTION	COMPLEX GAUSS CONTINUED FRACTION METHOD
.005	-.11459195770945768881171870D+05	-.11459195770945768881171870D+05
.010	-.57296503839534387814029990D+04	-.57296503839534387814029990D+04
.050	-.11461929533880912255798930D+04	-.11461929533880912255798930D+04
.100	-.57343946311317633105893220D+03	-.57343946311317633105893220D+03
.250	-.23014563670988900106772980D+03	-.23014563670988900106772980D+03
.500	-.11614982435180295584522110D+03	-.11614982435180295584522110D+03
.750	-.78407963278666147796055460D+02	-.78407963278666147796055460D+02
1.000	-.59671851253427766672521210D+02	-.59671851253427766672521210D+02
5.000	-.14122993109606857135954470D+02	-.14122993109606857135954470D+02
10.000	-.26651079459708560980737880D+01	-.26651079459708560980737880D+01

APPENDIX B

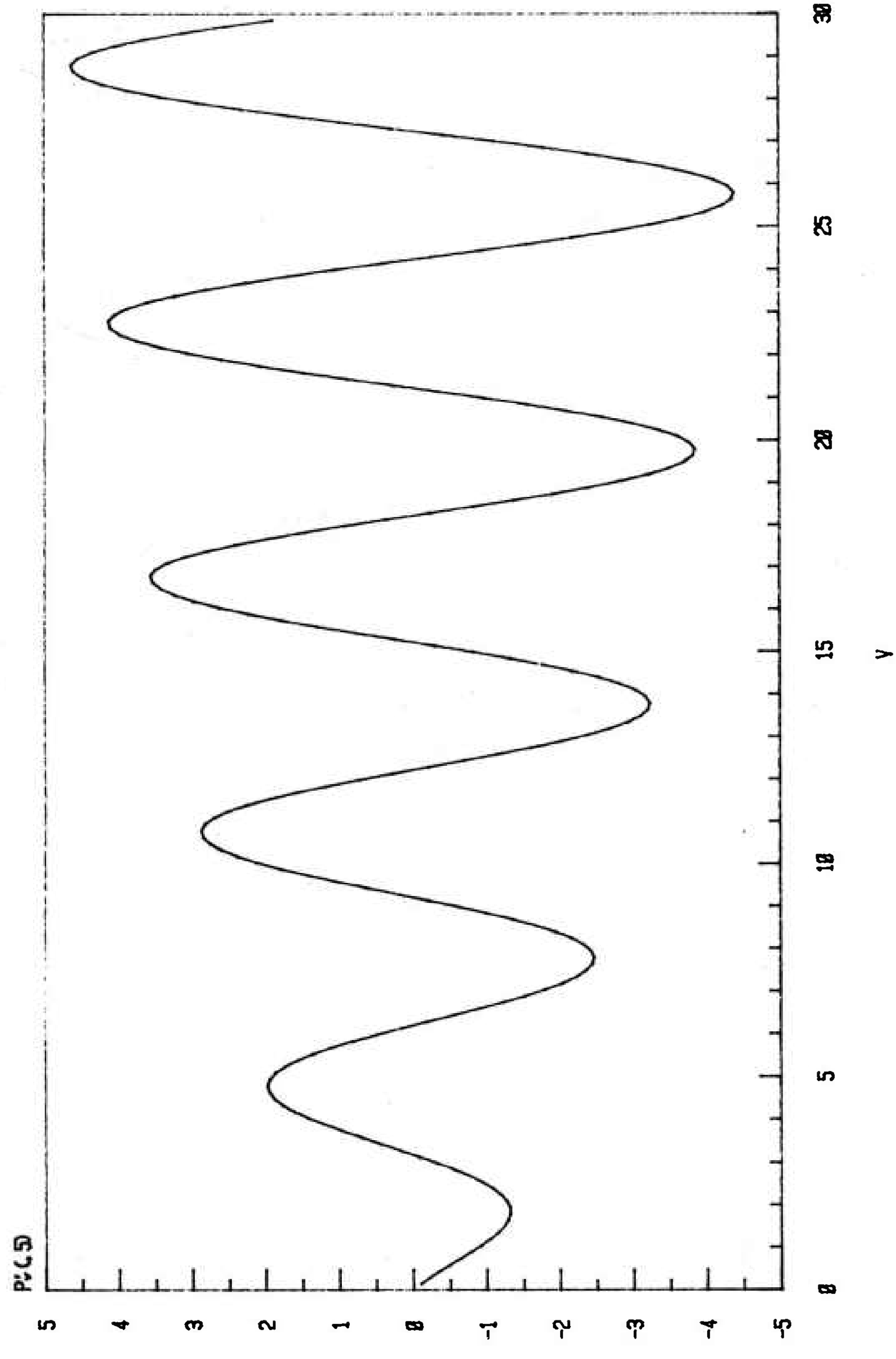
GRAPHS


P0V(.5) VS. V
P0V(.5)
V
1.0
0.5
0.0
-0.5
0
5
10
15
20
25
30
35
40
45
50

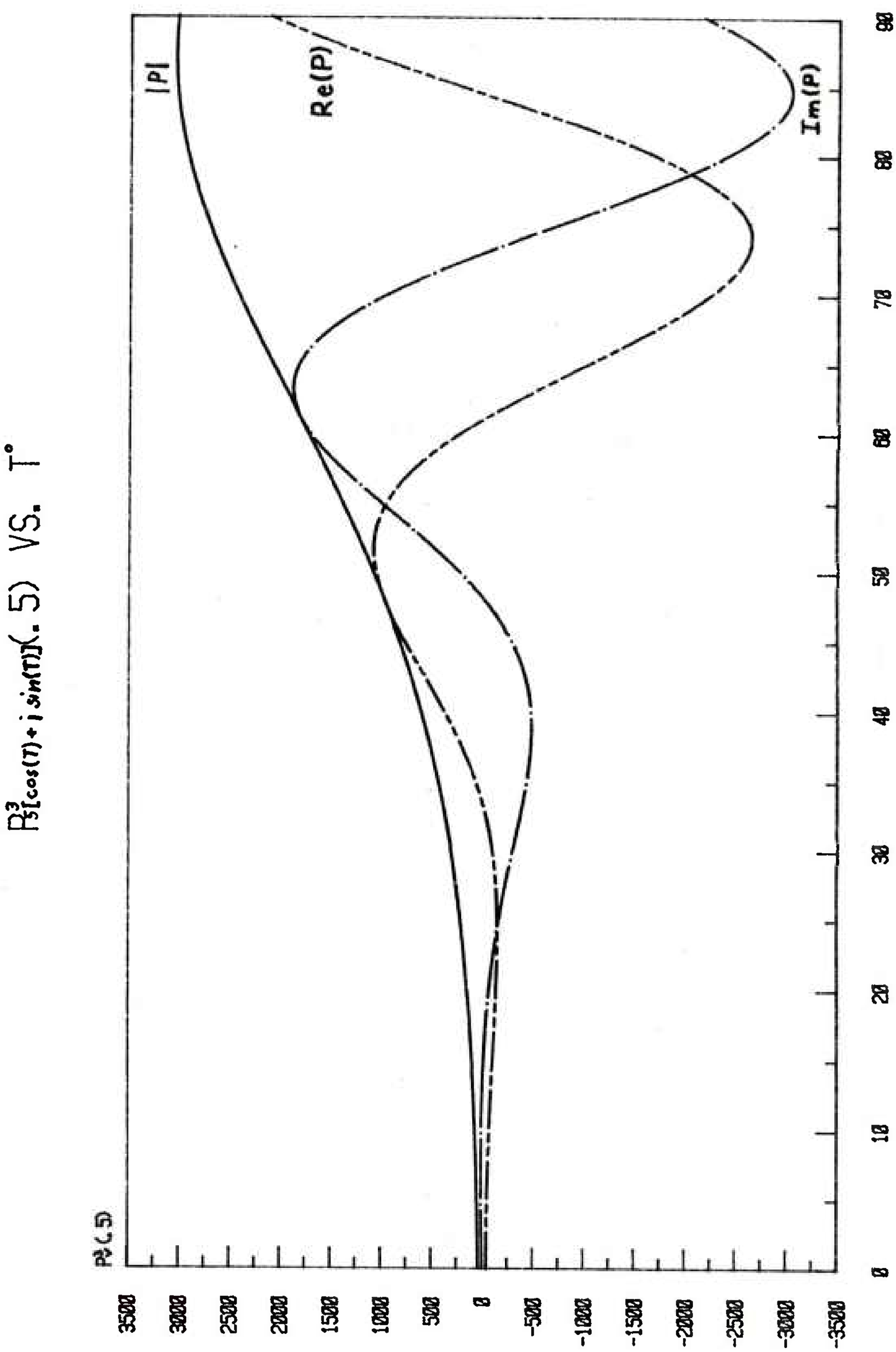

Q0v(.5) VS. V
Q0v(.5)
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
0
5
10
15
20
25
30
35
40
45
50
V



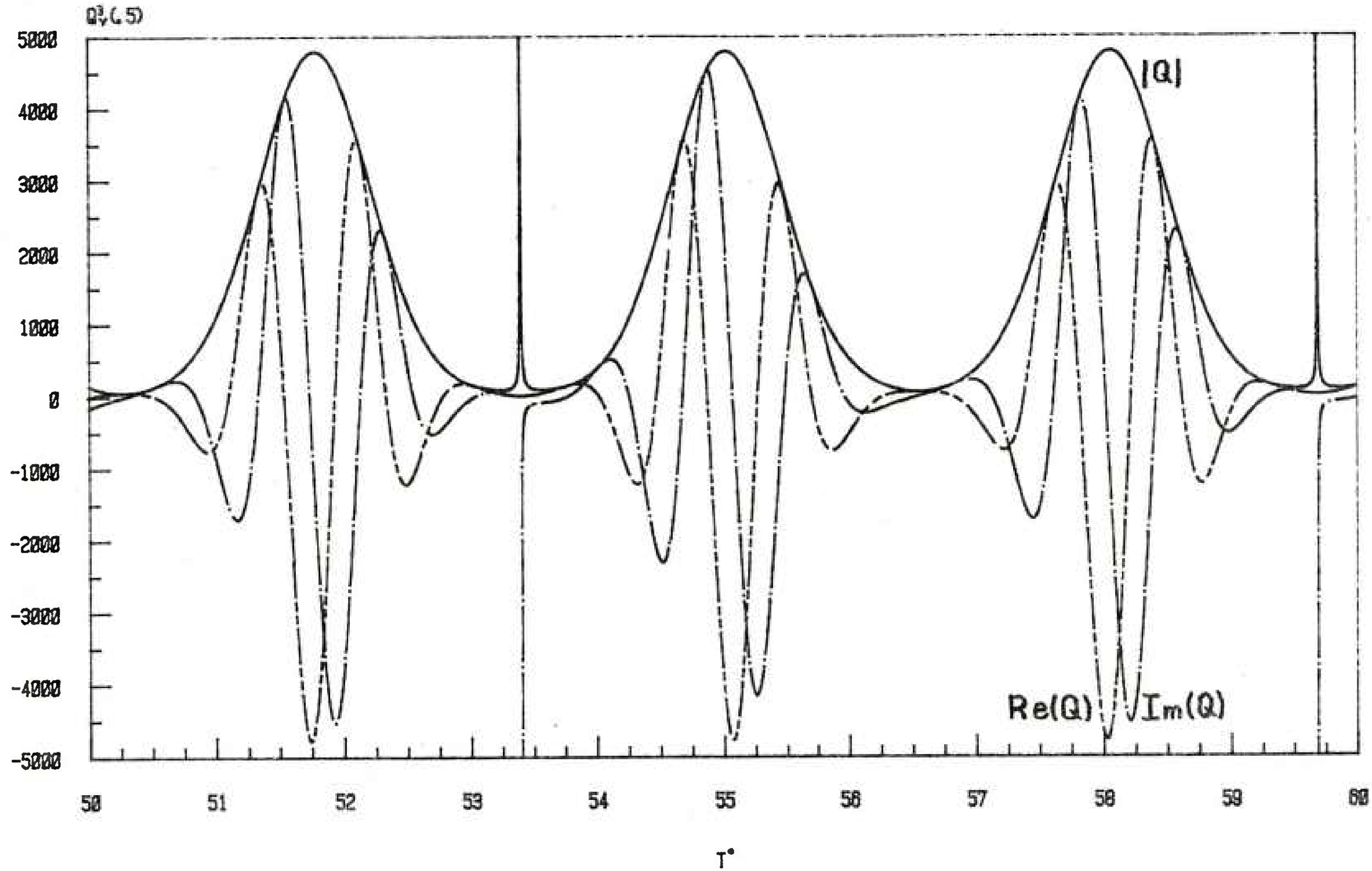
Q¹ᵥ(.5) VS. V
V
Q(.5)

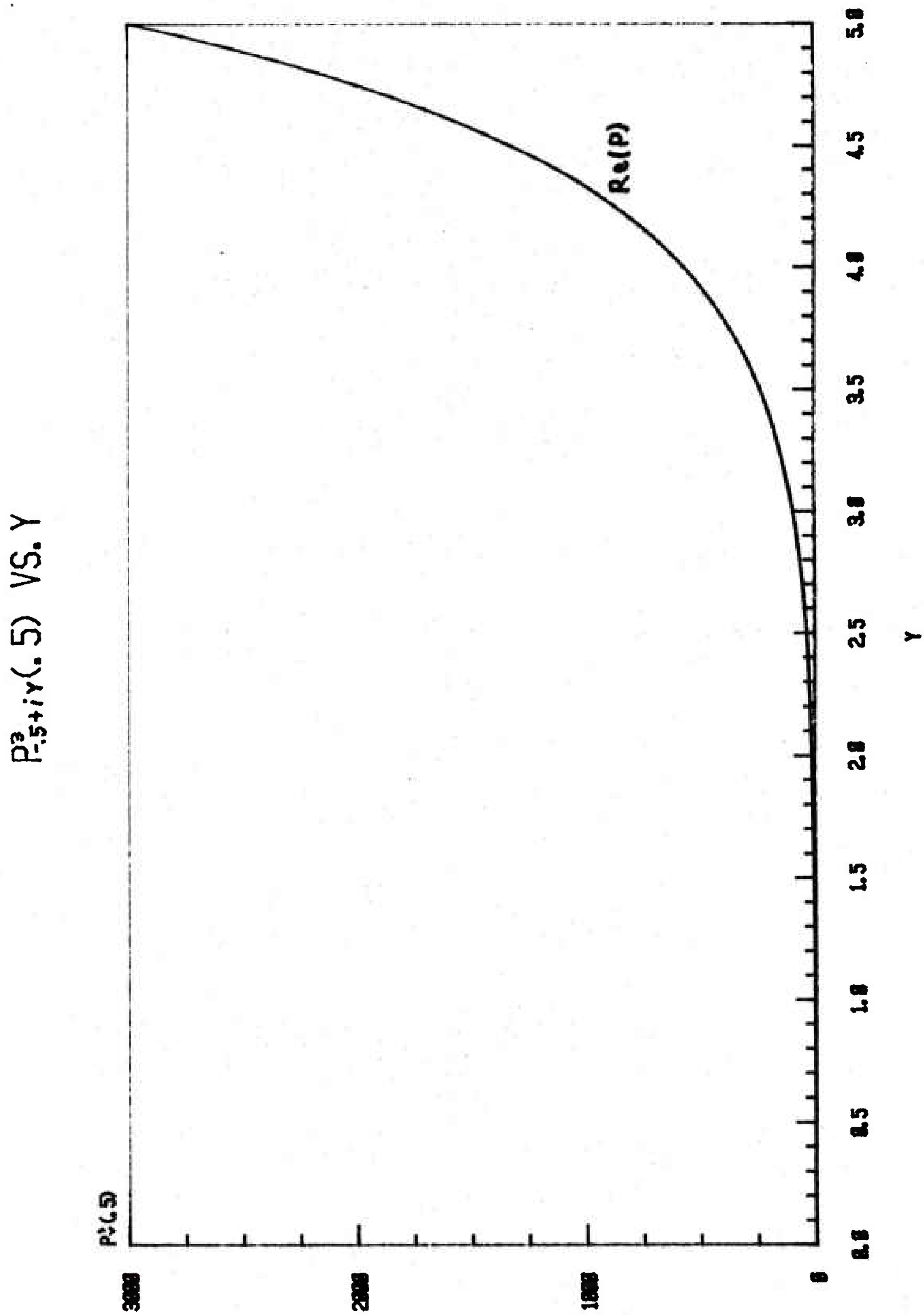


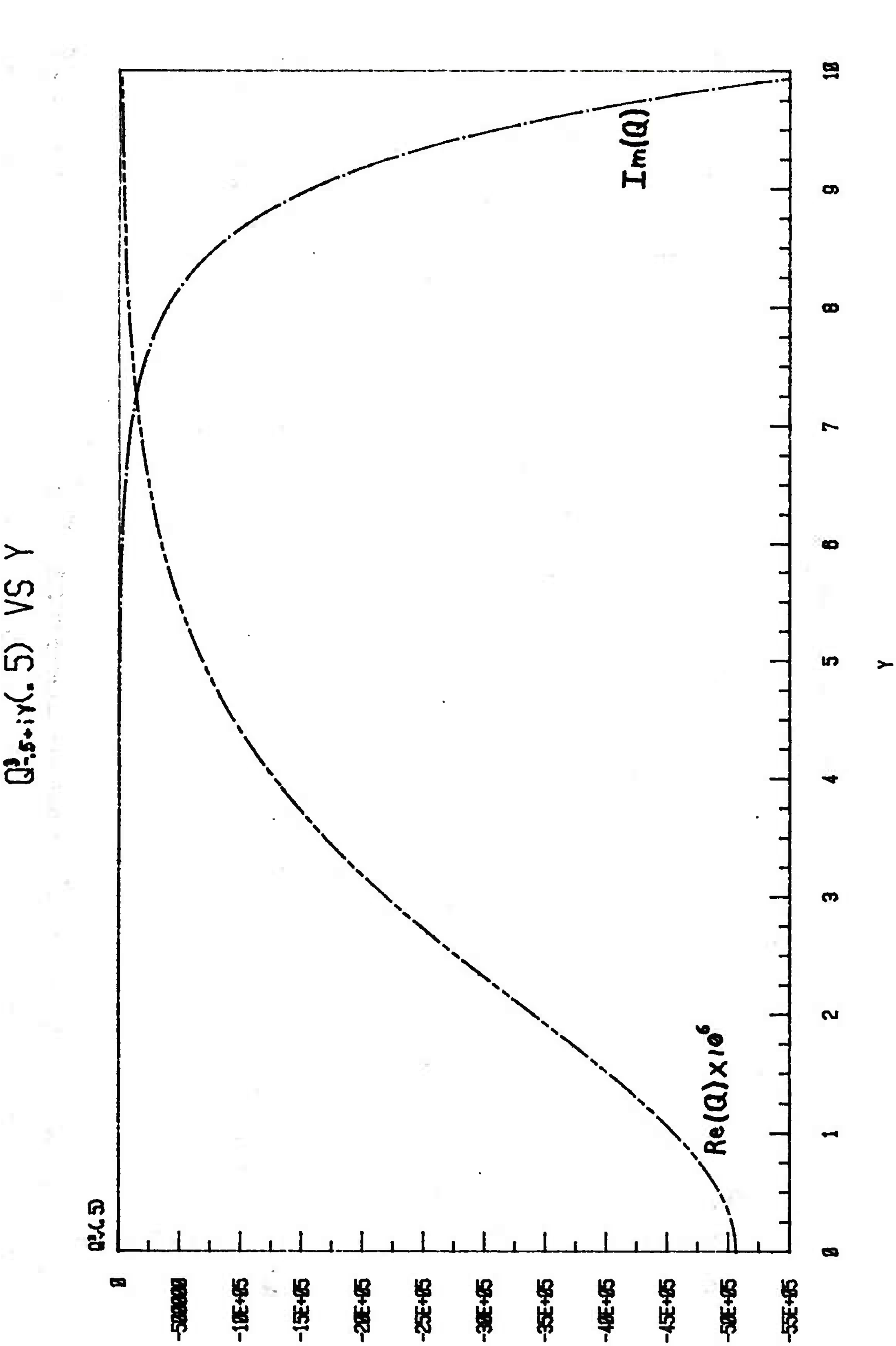
$P_\nu^1(.5)$ VS. V


$P^3_{5[\cos(T)+j\sin(T)]}(.5)$ VS. T°
|P|
Re(P)
Im(P)
$P^3_5(.5)$
3500
3000
2500
2000
1500
1000
500
0
-500
-1000
-1500
-2000
-2500
-3000
-3500
0
10
20
30
40
50
60
70
80
90
T°

$Q^3_{\nu[\cos(T)+i\sin(T)]}(.5)$ VS. T°




P³₋.₅₊ᵢʏ(.5) VS. Y
P³(.5)
3000
2000
1000
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
Y
Re(P)


$Q^{3}_{-.5+iY}(.5)$ VS Y
$Q^{3}(.5)$
Im(Q)
$Re(Q) \times 10^{6}$
Y
0
1
2
3
4
5
6
7
8
9
10
0
-500000
-10E+05
-15E+05
-20E+05
-25E+05
-30E+05
-35E+05
-40E+05
-45E+05
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-55E+05

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